

AD-A261 028

ATION PAGE

Form Approved
OMB No. 0704-0188

2



to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302, and to the Office of Management and Budget, Paperwork Reduction Project (0704-0188), Washington, DC 20503.

1. AGENCY USE ONLY (Leave blank)		2. REPORT DATE 1/18/93		3. REPORT TYPE AND DATES COVERED Technical	
4. TITLE AND SUBTITLE Variational Theory of Motion of Curved, Twisted and Extensible Elastic Rods				5. FUNDING NUMBERS DAAL03-92-6-0123	
6. AUTHOR(S) Iradj Tadjbakhsh and Dimitris C. Lagoudas				DTIC ELECTE FEB 22 1993 S C D	
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) Rensselaer Polytechnic Institute Dept. of Civil & Env. Engineer, JEC 4049 110 - 8th Street Troy, NY 12180-3590					
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES) U. S. Army Research Office P. O. Box 12211 Research Triangle Park, NC 27709-2211				10. SPONSORING/MONITORING AGENCY REPORT NUMBER ARO 30378.6-EG-URI	
11. SUPPLEMENTARY NOTES The view, opinions and/or findings contained in this report are those of the author(s) and should not be construed as an official Department of the Army position, policy, or decision, unless so designated by other documentation.					
12a. DISTRIBUTION/AVAILABILITY STATEMENT Approved for public release; distribution unlimited.				12b. DISTRIBUTION CODE	
13. ABSTRACT (Maximum 200 words) The variational theory of three dimensional motion of curved twisted and extensible elastic rods is obtained based entirely on the kinematical variables of position and rotations. The constitutive relations that define the resistive couples and the axial force as gradients of the strain energy function are established. A candidate for the strain energy function, derived on the basis of classical assumptions, is presented.					
14. SUBJECT TERMS				15. NUMBER OF PAGES 14	
				16. PRICE CODE	
17. SECURITY CLASSIFICATION OF REPORT UNCLASSIFIED		18. SECURITY CLASSIFICATION OF THIS PAGE UNCLASSIFIED		19. SECURITY CLASSIFICATION OF ABSTRACT UNCLASSIFIED	
				20. LIMITATION OF ABSTRACT UL	

302/100
93-03527
1508

VARIATIONAL THEORY OF MOTION OF CURVED, TWISTED AND EXTENSIBLE ELASTIC RODS

by

Iradj Tadjbakhsh
Department of Civil & Environmental Engineering
Rensselaer Polytechnic Institute
Troy New York 12180-3590

and
Dimitris C. Lagoudas
Department of Aerospace Engineering
Texas A&M University
College Station, Texas, 77843-3141

Accession For	
NTIS	CRA&I <input checked="" type="checkbox"/>
DTIC	TAB <input type="checkbox"/>
Unannounced <input type="checkbox"/>	
Justification	
By	
Distribution /	
Availability Codes	
Dist	Avail and/or Special
A-1	

100% QUALITY INSPECTED 3

The variational theory of three dimensional motion of curved twisted and extensible elastic rods is obtained based entirely on the kinematical variables of position and rotations. The constitutive relations that define the resistive couples and the axial force as gradients of the strain energy function are established. A candidate for the strain energy function, derived on the basis of classical assumptions, is presented.

INTRODUCTION

In the Historical Introduction of the "A Treatise on the Mathematical Theory of Elasticity," Love (1892) narrates that in 1742 Daniel Bernoulli wrote to Euler suggesting that the differential equation of the elastica could be found by making the integral of the work done or the square of the curvature a minimum. Acting on this suggestion Euler was able to obtain the differential equation of the elastica and the various forms of it. Thus the concept of the strain energy was born and the foundation of the variational theory of elastic rods were laid out. The equilibrium equations that were much later developed by Love are applicable to an initially bent and twisted rod.

Our aim in this paper is to establish a variational formulation for the title problem and in the process infer the existence of the strain energy function and determine the constitutive relations that relate this function to the bending and twisting couples and the axial force within the rod. This development together with equations of motion and the geometry of deformation define a direct approach and an exact nonlinear theory for the three dimensional motion of a one dimensional elastic medium capable of resisting bending twisting and extension. Going a step further, in order to actually construct an explicit form for the strain energy function, we enter the realm of hypothesis and use Kirchhoff's description of deformations in a thin rod. This view enables us to determine a strain energy function that can be used in engineering applications.

The recent history of investigations of the rod theories consists of developments along two separate streams, the direct approach and approximations from three-dimensional continuum. In the direct approach a one-dimensional continuum view is pursued and the medium is supposed endowed, in addition to its position, with vector fields, the directors,

that are to be interpreted appropriately to define bending, twist and extension properties of a rod. This approach has its origin in the work of E. and F. Cosserat (1909) and numerous investigations have contributed to it among them Naghdi (1982), Naghdi and Rubin (1984), Whitman and DeSilva (1970), Green and Laws (1966), and Ericksen (1970). Extensive investigation into the qualitative aspects of the nonlinear theory such as questions of existence of solutions and global behavior have been carried out by Antman (1976). His basic work entitled "The Theory of Rods" (1972) describes these theories both as approximations to the three-dimensional continuum theory and as a one-dimensional continuum with directors. The work presented here, although pertains to a one-dimensional continuum does not use directors, but is formulated entirely on the basis of kinematical quantities consisting of the position vector of points along the curve of centroids and the orientation angles of the cross sections of the rod relative to a fixed coordinate system. It is a generalization of the work of Tadjbakhsh (1966) in which the theory of planar motion of the extensible elastica was described.

The history of construction of approximate theories in the context of three-dimensional nonlinear continuum theory is also varied and to it many investigators including some of the above authors have contributed, see for example, Naghdi and Wenner (1974).

KINEMATICS

An elastica is a nearly uniform slender rod of finite length. In the unstressed state the centroids of the cross section form a space curve C that is called the reference curve with an arc length s . The orientation of the principal axes of the cross section vary continuously along the rod. This means that in the unstrained state the rod has arbitrary twist and curvatures. With respect to an inertial Cartesian frame \mathbf{x} the position of a point s in the unstrained state is denoted by $\mathbf{X} = X_i(s)\mathbf{n}_i$, $i = 1, 2, 3$, with \mathbf{n}_i being the dextral unit vectors of the frame \mathbf{x} .

The cross sectional area can be slowly varying function of s and will be denoted by $A(s)$. As the rod deforms the curve C acquires new configuration c that changes with time. The arc length along c is denoted by ξ that depends upon s and t , i.e. $\xi = \xi(s, t)$. The position of a point s on c at an arbitrary time is $\mathbf{x}(s, t)$ so that

$$\mathbf{x}(s, t_0) = \mathbf{X}(s), \quad \xi(s, t_0) = s \quad (1a, b)$$

where t_0 is a reference time at which the rod is in the unstrained state. Also

$$\frac{\partial \xi}{\partial s} = \xi' = (\mathbf{x}'_i \mathbf{x}'_i)^{\frac{1}{2}} \quad (2)$$

where prime denotes differentiation with respect to time and summation over a repeated index is implied. The strain e is defined by

$$e = \xi' - 1 \quad (3)$$

where $e > 0$ denotes extension and $e < 0$ contraction. The strict positivity of ξ' implies that $-1 \leq e < \infty$.

Attached to any point s of c a Cartesian coordinate frame y will be assumed and will be referred to as the body reference frame. The coordinate axes of the body frame are y_1, y_2, y_3 with the y_3 axis pointing in the direction of increasing s and y_1 and y_2 being the principal axes of the cross section. The dextral unit vectors of the y frame will be denoted by $e_i, i = 1, 2, 3$, with their orientations at the reference time t_0 being E_i .

Denoting by $l_{ij}(s, t)$ and $L_{ij}(s)$ the elements of the matrices of direction cosines of the dextral sets e_i and E_i one has

$$n_i = l_{ij} e_j = L_{ij} E_j \quad (4)$$

and

$$e_i = l_{ij} n_j, \quad E_i = L_{ij} n_j \quad (5a, b)$$

The angles φ_i represent rotations between the corresponding pairs of E_i and e_i when these directions are assumed to issue from a common origin, Fig. 1. These angles are determined through

$$\cos \varphi_i = e_i \cdot E_i = l_{ji} L_{ji}, \quad i=1, 2, 3 \text{ (sum only on } j) \quad (6)$$

The direction cosines l_{ij} are characterized non-uniquely by three orientations angles $\theta_1(s, t), \theta_2(s, t)$ and $\theta_3(s, t)$. These angles can be selected in a variety of ways and represent three finite rotations about unit vectors e_i or n_i . If these rotations are properly selected the fixed orientation n_i may be brought to any arbitrary body orientation e_i . Kane et al. (1983) list at least 24 possibilities for the order of rotations of the angles $\theta_1, \theta_2, \theta_3$ about the body set of unit vectors e_i or the fixed set of unit vectors n_i .

Regardless of the particular choice of orientation angles the angular velocity $\omega = \omega_i e_i$ of a cross sectional element Ad_s , Fig. 1 of the rod is determined uniquely from

$$\omega_i = \eta_{igh} \dot{l}_{ig} l_{ih}, \quad (\dot{} = \frac{\partial}{\partial t}) \quad (7)$$

where $\eta_{ijk} = \epsilon_{ijk} (\epsilon_{ijk} + 1)/2$, (no sum on i, j, k) and ϵ_{ijk} is the alternator tensor with the non-zero components $\epsilon_{123} = \epsilon_{231} = \epsilon_{321} = +1$ and $\epsilon_{132} = \epsilon_{321} = \epsilon_{213} = -1$.

The curvature vector $K = K_i e_i$ of the rod can be defined in a similar way with K_3 representing twist and K_1 and K_2 representing bending curvatures about the principal directions of the cross section. Using the dynamical analogy of E.I. Routh, Love (1944) has noted that if the frame y were to move with unit speed along the curve c such that at any point ξ of c it has the orientation of the y frame at that point then the angular velocities ω_1 and ω_2 will be the principal curvatures K_1 and K_2 of the rod. Also the angular velocity ω_3

will be the twist curvature K_3 of the rod. Thus the formulas that define the angular velocities from direction cosines can be used to determine curvatures, provided time differentiation is replaced by differential with respect to ξ . Therefore one has

$$K_i = \eta_{igh} \frac{\partial l_{ig}}{\partial \xi} l_{ih} = \frac{1}{1+e} \eta_{igh} l'_{ig} l_{ih} \equiv \frac{k_i}{1+e}, \quad (8)$$

where differentiation with respect to ξ has been replaced with differentiation with respect to s and curvature parameters $k_i = (1+e)K_i$ is also introduced. For future use one may note the formulas for derivatives of direction cosines l_{ij} and the unit vectors e_i .

$$l_{ij} = (1+e) \epsilon_{ghj} l_{ig} k_h \quad \dot{l}_{ij} = \epsilon_{ghj} l_{ig} w_h \quad (9a,b)$$

$$\dot{e}_i = \epsilon_{kji} k_j e_k \quad \dot{e}_i = \epsilon_{kji} w_j e_k \quad (10a,b)$$

if K_i be the curvatures of the rod in the unstrained state ($e = 0$) then from (8)

$$K_i = \eta_{igh} L'_{ig} L_{ih} \quad (11)$$

Since e_3 and $\partial \mathbf{x} / \partial \xi$ are both unit tangents to the central line one has

$$\mathbf{x}'_i = (1+e) l_{i3} \quad (12)$$

We assume that the center of mass of the cross sections coincide with the centroids. The linear and the central angular momentum per unit length are then given by

$$\mathbf{p} = \rho A \mathbf{x}_i \dot{\mathbf{x}}_i \quad (13)$$

and

$$\mathbf{H} = \rho I \omega = \rho (I_{11} \omega_1 \mathbf{e}_1 + I_{22} \omega_2 \mathbf{e}_2 + I_{33} \omega_3 \mathbf{e}_3) \quad (14)$$

where ρ is the mass density per unit unstrained length and I is the diagonal moment of inertia tensor with components

$$I_{11} = \int_A y_2^2 dA, \quad I_{22} = \int_A y_1^2 dA, \quad I_{33} = I_{11} + I_{22} \quad (15)$$

Equations of Motion

Referring to the body set of axes e_i one can define the vector \mathbf{F} of the resultant shear stresses F_1 and F_2 and the axial stress resultant F_3 . Similarly, one may define the couple stress vector \mathbf{M} consisting of the bending moments M_1 and M_2 and the torque M_3 . Explicitly we have

$$\mathbf{F} = F_i \mathbf{e}_i \quad \text{and} \quad \mathbf{M} = M_i \mathbf{e}_i \quad (16)$$

The well known dynamic equilibrium of the rod can be expressed by the equations of the balance of linear momentum

$$\mathbf{F}' + \mathbf{f} = \dot{\mathbf{p}} \quad (17)$$

and of the balance of angular momentum

$$\mathbf{M}' + \mathbf{x}' \times \mathbf{F} + \mathbf{m} = \dot{\mathbf{H}} \quad (18)$$

wherein \mathbf{f} and \mathbf{m} represent distributed force and moment acting on the rod. The scalar components of these equations can be referred to the body set of axes. For this purpose one needs to express all vector quantities in terms of unit vectors \mathbf{e}_i and use (9)–(10). Then (17) becomes

$$F'_1 + k_2 F_3 - k_3 F_2 + \dot{f}_1^y = \rho A \bar{x}_j l_{j1} \quad (19a)$$

$$F'_2 + k_3 F_1 - k_1 F_3 + \dot{f}_2^y = \rho A \bar{x}_j l_{j2} \quad (19b)$$

$$F'_3 + k_1 F_2 - k_2 F_1 + \dot{f}_3^y = \rho A \bar{x}_j l_{j3} \quad (19c)$$

while (18) assumes the form

$$M'_1 + k_2 M_3 - k_3 M_2 - (1+e)F_2 + \dot{m}_1^y = \rho I_1 (\omega_1 + \omega_2 \omega_3) \quad (20a)$$

$$M'_2 + k_3 M_1 - k_1 M_3 - (1+e)F_1 + \dot{m}_2^y = \rho I_2 (\omega_2 + \omega_1 \omega_3) \quad (20b)$$

$$M'_3 + k_1 M_2 - k_2 M_1 + \dot{m}_3^y = \rho [J \dot{\omega}_3 + (I_2 - I_1) \omega_1 \omega_2] \quad (20c)$$

where $I_1 = I_{11}$, $I_2 = I_{22}$ and $I = J_{33} = I_1 + I_2$. The superscript y on the components of \mathbf{f} and \mathbf{m} denote the components of these vectors in the body reference frame.

To express the equations of motion in the inertial frame we introduce the components of the stress resultants in that frame. Thus

$$\mathbf{F}_i^x = l_{ij} F_j \quad \mathbf{M}_i^x = l_{ij} M_j \quad (21a,b)$$

Then (19) becomes

$$F_i^x + \dot{f}_i^x = \rho A \bar{x}_i \quad (22)$$

and (20) assumes the form

$$M_1^x - (1+e)F_2 + \dot{m}_1^x = \rho (I_{rj} l_{ij} \dot{\omega}_r + \epsilon_{grj} I_{sj} l_{ig} \omega_s \omega_r) \quad (23a)$$

$$M_2^x - (1+e)F_1 + \dot{m}_2^x = \rho (I_{rj} l_{2j} \dot{\omega}_r + \epsilon_{grj} I_{sj} l_{2g} \omega_s \omega_r) \quad (23b)$$

$$M_3^{x'} + m_3^x = \rho(I_{rj} l_{3j} \dot{\omega}_r + \epsilon_{grj} I_{sj} l_{3g} \omega_s \omega_r) \quad (23c)$$

Either of the set of equations (19)–(20) or (22)–(23) can be considered as the governing differential equations of motion. These equations will have to be supplemented with constitutive relations that define resultant axial stress F_3 and the resultant bending and twisting couples M_1, M_2, M_3 in terms of the axial strain e and curvatures k_1, k_2, k_3 . In the next section we consider the derivation of these constitutive relations.

Constitutive Relations

We assume that the motion of the elastic rod is equivalent to the stationarity of the Hamiltonian H which is defined by

$$\begin{aligned} H[l_{ij}(\varphi), x_i, e] &= \int_{t_1}^{t_2} \int_{s_1}^{s_2} \mathcal{L} ds dt + \\ &\int_{t_1}^{t_2} \left[\bar{F}_i \bar{X}_i + \bar{M}_i \bar{\varphi}_i \right]_{s_1}^{s_2} dt - \int_{s_1}^{s_2} \left[\rho A \bar{v}_i x_i \right]_{t_1}^{t_2} ds - \\ &\int_{s_1}^{s_2} \rho \left[I_1 \bar{\omega}_1 \varphi_1 + I_2 \bar{\omega}_2 \varphi_2 + J \bar{\omega}_3 \varphi_3 \right]_{t_1}^{t_2} ds \end{aligned} \quad (24)$$

where \mathcal{L} is the action density function

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} \rho A \dot{x}_i \dot{x}_i + \frac{1}{2} \rho (I_1 \dot{\omega}_1^2 + I_2 \dot{\omega}_2^2 + J \dot{\omega}_3^2) - w(e, k_i) + \\ &\lambda_i [l_{i3} (1+e) - x_i'] + f_i^x x_i + m_i^y \varphi_i \end{aligned} \quad (25)$$

and \bar{F}_i^1, \bar{M}_i^1 and \bar{F}_i^2, \bar{M}_i^2 are the applied forces and moments at ends s_1 and s_2 respectively.

Also $\bar{v}_i^{1,2}$ and $\bar{\omega}_i^{1,2}$ are the initial and final linear and angular velocities. The strain energy function w depends upon the kinematical variables e and k_i . The precise nature of this dependence is the constitutive relations that we seek and is a consequence of the stationarity of H . The functions λ_i are the Lagrange multipliers that allow the constraints (12) to be incorporated within the Hamiltonian. As a result x_i and l_{ij} can be regarded as independent variables. Additionally the constraint (12) implies the definition (2)–(3) for the strain e and hence in (24) e can also be viewed as an independent variable. To see this we need to note that if each side of (12) is multiplied by itself we obtain $x_i' x_i' =$

$(1+e)^2 l_{i3} l_{i3} = (1+e)^2$ which is restatement of (2)–(3). The terms $f_i^x x_i$ and $m_i^y \varphi_i$ in (25) represent the density of the potential of the applied forces and moments on the rod. As stated in (5) the angles φ_i are the rotations from E_i to e_i .

With these preliminaries we note that the Euler equation corresponding variations δx_i is simply $\lambda'_i + \dot{f}_i^x = \rho A \ddot{x}_i$ which when compared with (22) reveals that $\lambda_i = F_i^x$. Next considering the variations with respect to e we obtain

$$\frac{\partial W}{\partial e} = F_i^x l_{i3} = F_3 \quad (26)$$

which is the constitutive relationship determining the axial force F_3 as the derivative of strain energy with respect to axial strain e .

We now turn to the Euler equation corresponding to the variation $\delta \varphi_1$. For this purpose we note that l_{ij} , ω_i and k_i depend on φ_1 . For orientation angles of the cross section we select the sequence of body rotations first $\theta_2 e_2$, second $\theta_3 e_3$ and the third $\theta_1 e_1$ with $\theta_1 \equiv \varphi_1$. In this sequence the last rotation is through φ_1 with respect to which variation is sought. The matrix l of the direction cosines is given by

$$l = B(\theta_2)C(\theta_3)A(\theta_1) = \begin{bmatrix} C_2 C_3 & -C_1 C_2 S_3 + S_1 S_2 & S_1 C_2 S_3 + C_1 S_2 \\ S_3 & C_1 C_3 & -S_1 C_3 \\ -S_2 C_3 & C_1 S_2 S_3 + S_1 C_2 & -S_1 S_2 S_3 + C_1 C_2 \end{bmatrix} \quad (27)$$

where

$$C_i = \cos \theta_i, \quad S_i = \sin \theta_i \quad (28)$$

$$A(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix} \quad B(\theta) = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}$$

$$C(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (29)$$

Subsequently, we find from (7) and (8)

$$\omega_1 = \dot{\theta}_2 S_3 + \dot{\theta}_1, \quad \omega_2 = \dot{\theta}_2 C_1 C_3 + \dot{\theta}_3 S_1, \quad \omega_3 = -\dot{\theta}_2 S_1 C_3 + \dot{\theta}_3 C_1 \quad (30)$$

$$k_1 = \dot{\theta}_2' S_3 + \dot{\theta}_1', \quad k_2 = \dot{\theta}_2' C_1 C_3 + \dot{\theta}_3' S_1, \quad k_3 = -\dot{\theta}_2' S_1 C_3 + \dot{\theta}_3' C_1 \quad (31)$$

From (24) we have

$$\left\{ -\frac{\partial W}{\partial \mathbf{k}_1} \frac{\partial \mathbf{k}_1}{\partial \varphi_1} + \frac{\partial}{\partial s} \left(\frac{\partial W}{\partial \mathbf{k}_1} \frac{\partial \mathbf{k}_1}{\partial \varphi_1} \right) + (1+e) F_1^x \left[\frac{\partial l_{13}}{\partial \varphi_1} - \frac{\partial}{\partial s} \left(\frac{\partial l_{13}}{\partial \varphi_1} \right) \right] \right\} + m_1^y = \frac{\partial}{\partial t} \left[\rho I_1 \omega_1 \frac{\partial \omega_1}{\partial \varphi_1} \right] - \rho I_2 \omega_2 \frac{\partial \omega_2}{\partial \varphi_1} - \rho J \omega_3 \frac{\partial \omega_3}{\partial \varphi_1} \quad (32)$$

Noting that $\theta_1 \equiv \varphi_1$ we have from (30) $\partial \omega_1 / \partial \varphi_1 = 1$, $\partial \omega_2 / \partial \varphi_1 = \omega_3$, $\partial \omega_3 / \partial \varphi_1 = -\omega_2$. Also from (31) $\partial \mathbf{k}_1 / \partial \varphi_1 = 0$, $\partial \mathbf{k}_2 / \partial \varphi_1 = \mathbf{k}_3$, $\partial \mathbf{k}_3 / \partial \varphi_1 = -\mathbf{k}_2$ and from (27) $\partial l_{13} / \partial \varphi_1 = -l_{12}$. Using these results and the inverse of (21a), (32) becomes

$$\left(\frac{\partial W}{\partial \mathbf{k}_1} \right)' + k_2 \frac{\partial W}{\partial \mathbf{k}_3} - k_3 \frac{\partial W}{\partial \mathbf{k}_2} - (1+e) F_2 + m_1^y = \rho I_1 (\dot{\omega}_1 + \omega_2 \omega_3) \quad (33)$$

In exactly the same manner one can proceed to determine the Euler equation for a variation $\delta \varphi_2$. Now the consecutive sequence of body rotations $\theta_3 \mathbf{e}_3$, $\theta_1 \mathbf{e}_1$ and $\theta_2 \mathbf{e}_2$ is selected with $\theta_2 \equiv \varphi_2$. Without going into details one obtains

$$\left(\frac{\partial W}{\partial \mathbf{k}_2} \right)' + k_3 \frac{\partial W}{\partial \mathbf{k}_1} - k_1 \frac{\partial W}{\partial \mathbf{k}_3} - (1+e) F_1 + m_2^y = \rho I_2 (\dot{\omega}_2 + \omega_1 \omega_3) \quad (34)$$

For variation of φ_3 we adopt the consecutive sequence of body rotations $\theta_1 \mathbf{e}_1$, $\theta_2 \mathbf{e}_2$ with $\theta_3 \equiv \varphi_3$. The matrix of direction cosines is

$$l = A(\theta_1) B(\theta_2) C(\theta_3) = \begin{bmatrix} C_2 C_3 & -C_2 S_3 & S_2 \\ S_1 S_2 C_3 + C_1 S_3 & -S_1 S_2 S_3 + C_1 C_3 & -S_1 C_2 \\ -C_1 S_2 C_3 + S_1 S_3 & C_1 S_2 S_3 + S_1 C_3 & C_1 C_2 \end{bmatrix} \quad (35)$$

with angular velocities of the cross section and the curvatures given by

$$\omega_1 = \dot{\theta}_1 C_2 C_3 + \dot{\theta}_2 S_3, \quad \omega_2 = \dot{\theta}_2 C_3 - \dot{\theta}_1 C_2 S_3, \quad \omega_3 = \dot{\theta}_3 + \dot{\theta}_1 S_2 \quad (36)$$

$$k_1 = \theta_1' C_2 C_3 + \theta_2' S_3, \quad k_2 = \theta_2' C_3 - \theta_1' C_2 S_3, \quad k_3 = \theta_3' + \theta_1' S_2 \quad (37)$$

For this case l_{13} does not depend on θ_3 and hence the Euler variational equation assumes the form

$$\left(\frac{\partial W}{\partial \mathbf{k}_3} \right)' + k_1 \frac{\partial W}{\partial \mathbf{k}_2} - k_2 \frac{\partial W}{\partial \mathbf{k}_1} + m_3^y = \rho [J \dot{\omega}_3 + (I_2 - I_1) \omega_1 \omega_2] \quad (38)$$

The specified boundary conditions at $s = s_1, s_2$ must be consistent with

$$[(\bar{F}_1 - F_1^x) \delta x_1 + (\bar{M}_1 - M_1) \delta \varphi_1]_{s_1}^{s_2} = 0 \quad (39)$$

for arbitrary and independent variations δx_1 and $\delta \varphi_1$. Similar restrictions are imposed on initial and final data, i.e.

$$\left\{ \rho A (\dot{x}_1 - \bar{v}_1) \delta x_1 + \rho [I_1 (\omega_1 - \bar{\omega}_1) \delta \varphi_1 + I_2 (\omega_2 - \bar{\omega}_2) \delta \varphi_2 + J (\omega_3 - \bar{\omega}_3) \delta \varphi_3] \right\}_{t_1}^{t_2} = 0 \quad (40)$$

Comparison of equations (33), (34) and (38) with equations (20a,b,c) respectively, establishes the constitutive relations

$$M_i = \frac{\partial W}{\partial k_i} \quad i = 1, 2, 3 \quad (41)$$

A STRAIN ENERGY FUNCTION

In order to gain an insight into the nature of the strain energy function we consider the strain of the lines and angles in the cross section of the rod. For this purpose we invoke the Kirchhoff hypothesis which assumes that the plane cross sections of rod that are normal to the axial direction in the unstrained state remain normal to the strained axial direction during deformation. Therefore the position vector to a material point in the cross-section before and after deformation can be given by

$$\mathbf{R} = \mathbf{X}(s) + y_1 \mathbf{E}_1(s) + y_2 \mathbf{E}_2(s) \quad (42)$$

and

$$\mathbf{r} = \mathbf{x}(s) + \alpha(s)[y_1 \mathbf{e}_1(s) + y_2 \mathbf{e}_2(s)] \quad (43)$$

respectively. The parameter $\alpha(s)$ is to be fixed by enforcing traction-free boundary conditions on the lateral surface of the rod.

Using the concept of extensional strains for stretching of line elements and distortion of angles between perpendicular lines as shear strains (Wempner, 1991), we define components of strain by

$$\epsilon_{ij} = \frac{1}{2}(\mathbf{g}_i \cdot \mathbf{g}_j - \mathbf{G}_i \cdot \mathbf{G}_j) \quad (44)$$

where

$$\begin{aligned} \mathbf{g}_1 &= \frac{\partial \mathbf{r}}{\partial y_1} = \alpha \mathbf{e}_1, \quad \mathbf{g}_2 = \frac{\partial \mathbf{r}}{\partial y_2} = \alpha \mathbf{e}_2, \\ \mathbf{g}_3 &= \frac{\partial \mathbf{r}}{\partial y_3} = \alpha' y_1 \mathbf{e}_1 + \alpha' y_2 \mathbf{e}_2 + \alpha y_1 \mathbf{e}_1' + \alpha y_2 \mathbf{e}_2' + (1 + \epsilon) \mathbf{e}_3 \end{aligned} \quad (45)$$

$$\mathbf{G}_1 = \frac{\partial \mathbf{R}}{\partial y_1} = \mathbf{E}_1, \quad \mathbf{G}_2 = \frac{\partial \mathbf{R}}{\partial y_2} = \mathbf{E}_2,$$

$$\mathbf{G}_3 = \frac{\partial \mathbf{R}}{\partial y_3} = y_1 \mathbf{E}_1' + y_2 \mathbf{E}_2' + \mathbf{E}_3 \quad (46)$$

Using (8) we can establish

$$\mathbf{e}_i' \cdot \mathbf{e}_j = \epsilon_{ijm} \mathbf{k}_m \quad (47)$$

$$\mathbf{E}_i' \cdot \mathbf{E}_j = \epsilon_{ijm} \mathbf{K}_m \quad (48)$$

where \mathbf{K}_i is the curvatures and twist in the unstrained state. Therefore (9) yields as the strain components

$$\begin{aligned} \epsilon_{11} &= \epsilon_{22} = \frac{1}{2}(\alpha^2 - 1), \quad \epsilon_{12} = 0 \\ \epsilon_{13} &= \frac{1}{2}[\alpha(\alpha' y_1 - \alpha y_2 k_3) + y_2 K_3] \\ \epsilon_{23} &= \frac{1}{2}[\alpha(\alpha' y_2 + \alpha y_1 k_3) - y_1 K_3] \\ \epsilon_{33} &= e + \frac{1}{2}e^2 - y_1[(1+e)\alpha k_2 - K_2] + y_2[(1+e)\alpha k_1 - K_1] - y_1 y_2(\alpha^2 k_1 k_2 - K_1 K_2) + \\ &\quad \frac{1}{2}y_1^2[\alpha'^2 + \alpha^2(k_2^2 + k_3^2) - K_2^2 - K_3^2] \\ &\quad + \frac{1}{2}y_2^2[\alpha'^2 + \alpha^2(k_1^2 + k_3^2) - K_1^2 - K_3^2] \end{aligned} \quad (49)$$

For a linear isotropic elastic material the non-zero stress components per unit strained area are

$$\begin{aligned} \sigma^{11} &= (\lambda + G)(\alpha^2 - 1) + \lambda \epsilon^{33}, \quad \sigma^{12} = 0 \\ \sigma^{22} &= (\lambda + G)(\alpha^2 - 1) + \lambda \epsilon^{33}, \quad \sigma^{23} = 2G \epsilon_{23} \\ \sigma^{33} &= \lambda(\alpha^2 - 1) + (\lambda + 2G)\epsilon^{33}, \quad \sigma^{31} = 2G \epsilon_{31} \end{aligned} \quad (50)$$

where λ and G are Lamé's constant and the shear modulus, respectively. The traction per unit undeformed area is then given by

$$\mathbf{t}^3 = \sigma^{3i} \mathbf{g}_i \quad (51)$$

One can define the axial stress resultant F_3 by

$$\begin{aligned} F_3 &= \int_A \mathbf{t}^3 \cdot \mathbf{e}_3 \, dA = \\ &= A(1+e) \left[(\lambda + 2G)(e + \frac{1}{2}e^2) + \lambda(\alpha^2 - 1) \right] \\ &\quad + (\lambda + 2G) \left[I_1[(1+e)\alpha k_1 - K_1]\alpha k_1 + I_2[(1+e)\alpha k_2 - K_2]\alpha k_2 \right. \\ &\quad \left. + \frac{1}{2}I_1(1+e)(\alpha'^2 + \alpha^2 k_1^2 - K_1^2 + \alpha^2 k_3^2 - K_3^2) \right. \\ &\quad \left. + \frac{1}{2}I_2(1+e)(\alpha'^2 + \alpha^2 k_2^2 - K_2^2 + \alpha^2 k_3^2 - K_3^2) \right] \end{aligned} \quad (52)$$

Similarly we have

$$M_1 = \int_A \alpha y_2 t^3 \cdot e_3 dA = \alpha(\lambda+2G)I_1 \left[(1+e)\alpha k_1 - K_1 \right] (1+e) \\ + I_1 \alpha^2 k_1 \left[\lambda(\alpha^2 - 1) + (\lambda + 2G)(e + \frac{1}{2}e^2) \right] \quad (53)$$

$$M_2 = - \int_A \alpha y_1 t^3 \cdot e_3 dA = \alpha(\lambda+2G)I_2 \left[(1+e)\alpha k_2 - K_2 \right] (1+e) \\ + I_2 \alpha^2 k_2 \left[\lambda(\alpha^2 - 1) + (\lambda + 2G)(e + \frac{1}{2}e^2) \right] \quad (54)$$

$$M_3 = - \int_A \alpha (y_1 t^3 \cdot e_3 - y_2 t^3 \cdot e_1) dA = JG\alpha^2(\alpha^2 k_3 - K_3) \\ + J\alpha^2 k_3 \left[\lambda(\alpha^2 - 1) + (\lambda + 2G)(e + \frac{1}{2}e^2) \right] \quad (55)$$

One may note that the integrability conditions

$$\frac{\partial F_3}{\partial k_1} = \frac{\partial M_1}{\partial e}, \quad \frac{\partial F_3}{\partial k_2} = \frac{\partial M_2}{\partial e}, \quad \frac{\partial F_3}{\partial k_3} = \frac{\partial M_3}{\partial e}, \\ \frac{\partial M_1}{\partial k_2} = \frac{\partial M_2}{\partial k_1}, \quad \frac{\partial M_1}{\partial k_3} = \frac{\partial M_3}{\partial k_1}, \quad \frac{\partial M_2}{\partial k_3} = \frac{\partial M_3}{\partial k_2} \quad (56)$$

are satisfied. Hence existence of a strain energy function is assured and by integration we have

$$W = A \frac{\lambda+2G}{2} e^2 (1+e+\frac{e^2}{4}) - A\lambda(1-\alpha^2)(e+\frac{1}{2}e^2) \\ + \lambda I_1 \alpha^2 (\alpha^2 - 1) \frac{k_1^2}{2} + (\lambda+2G)I_1 \alpha k_1 \left[\frac{1+2e^2+4e}{4} \alpha k_1 - (1+e)K_1 \right] \\ + \lambda I_2 \alpha^2 (\alpha^2 - 1) \frac{k_2^2}{2} + (\lambda+2G)I_2 \alpha k_2 \left[\frac{1+2e^2+4e}{4} \alpha k_2 - (1+e)K_2 \right] \\ + \frac{\lambda+2G}{2} (1+e)^2 \left[I_1 (\alpha'^2 + \alpha^2 k_1^2 - K_1^2) + I_2 (\alpha'^2 + \alpha^2 k_2^2 - K_2^2) \right] \\ + \lambda J \alpha^2 (\alpha^2 - 1) \frac{k_3^2}{2} + \frac{\lambda+2G}{2} J (e+\frac{1}{2}e^2) (\alpha^2 k_3^2 - K_3^2) + \frac{1}{2} G J \alpha (\alpha k_3 - K_3)^2 \quad (57)$$

For an initially straight rod K_1 should be set equal to zero. The above form of W reflects material isotropy, i.e. $W(e, k_1, k_2, k_3) = W(e, k_2, k_1, k_3)$, provided that $I_1 = I_2$.

We note that positive curvatures imply positive bending moments and conversely negative curvatures imply negative bending moments provided that $e > (-1 + 1/\sqrt{3})$ for $\alpha = 1$. This shows that equations (53) have a limited range of validity if the sense correspondence between moments and curvatures is to be retained. We also note the second order coupling between the squares of the curvatures and the axial strain e in (52), which implies that axial force can be generated by bending or twist only.

The parameter $\alpha(s)$ depends on the boundary conditions applied at the lateral surface of the rod. If the lateral surface is fixed, then $\alpha(s) = 1$. For zero tractions on the lateral surface, a condition appropriate for thin flexible rods is adopted according to which the average of σ^{11} and σ^{22} over the cross-section should vanish, i.e.

$$\int_A \sigma^{11} dA = \int_A \sigma^{22} dA = 0 \quad (58)$$

The above condition reduces to

$$\begin{aligned} H(\alpha, k_i, e) &\equiv \alpha' J + \alpha^2 (I_1 k_1^2 + I_2 k_2^2 + J k_3^2 + \frac{A}{\nu}) \\ &- (I_1 K_1^2 + I_2 K_2^2 + J K_3^2) + 2Ae(1 + \frac{1}{2}e) - \frac{A}{\nu} = 0 \end{aligned} \quad (59)$$

This equation should be interpreted as a differential equation for $\alpha(s)$ when k_i and e are known. To achieve this (52) is solved in an iterative procedure in which at every step k_i and e are known. We begin by writing (52) as $\lim_{n \rightarrow \infty} H(\alpha_{n+1}, k_i^n, e_n) = 0$ with $n = 0, 1, 2, 3, \dots$ For the first iteration ($n=0$), $e_0 = 0$, $k_i^0 = K_i$, $\alpha_1 = 1$, and the six equations (15) – (16) after using (22), (33) and (57) contain only six unknown quantities F_1 , F_2 , e_1 and k_i^1 when the externally applied forces and moments are prescribed. Solution of this set of equations enables one to use k_i^1 in the curvature-orientation angle relations such as (31) to determine the latter i.e. θ_i . Now l_{ij} (θ_k) are known and one proceeds to determine φ_i from (21) and x_i from (9) using the appropriate boundary conditions. The solution for the first iteration is complete. One enters (52) with e_1 and k_i^1 and computes α_2 and the iteration proceeds.

Acknowledgement

This work was supported by the Army Research Office grant No. DAAL – 03 – 92 G – 0123 to Rensselaer Polytechnic Institute.

REFERENCES

Antman, S.S., "Ordinary Differential Equations of Non-Linear Elasticity I: Foundations of the Theories of Non-Linearly Elastic rods and Shells," A.R.M.A. 61 (1976), 307-351.

Antman, S.S., "The Theory of Rods", Handbuch der Physik, Vol. VIa/2, Springer-Verlag, Berlin, 1972.

Cosserat, E. and F., "The Des Corps Deformables, A. Hermann et Fils, Paris, 1909.

Ericksen, J.L., "Simpler Static Problems in Nonlinear Theories of Rods," Int. J. Solids Structures 6 (1970) 371-377.

Green, A.E. and L. Laws, "A General Theory of Rods", Proc. Royal Soc. Lond. A293 (1966) 145-155.

Green, A.E., P.M. Naghdi and M.L. Wenner, "On the Theory of Rods. I Derivations from the Three-Dimensional Equations and II Developments by Direct Approach," Proc. Royal Society London A337 (1974) 451-507.

Kane, T.R., Likins, P.W. and Levinson, D.A., "Spacecraft Dynamics," McGraw-Hill, New York, 1963.

Love, A.E.H., "A Treatise of the Mathematical Theory of Elasticity", Dover, 1944.

Naghdi, P.M. and M.B. Rubin, "Constrained Theories of Rods", J. Elasticity 14 (1984) 343-361.

Naghdi, P.M., "Finite Deformation of Elastic Rods and Shells," Proc. IUTAM Symp. - Finite Elasticity, Bethlehem, PA, 1980 (Edited by D.E. Carlson and R.T. Shield), 47-103, Martinus Nijhoff, The Hague, 1982.

Poynting, J.H., "On Pressure Perpendicular to the Shear Planes in Finite Pure Shears, and on the Lengthening of Loaded Wires When Twisted," Proc. Roy. Soc. London A82 (1909) 546-559.

Rivlin, R.S. and D.W. Saunders, "Large Elastic Deformations of Isotropic Materials. VII. Experiments on the Deformation of Rubber," Phil. Trans. Roy. Soc. London, Ser. A, No. 865, 243 (1951) 251-288.

Tadjbakhsh, I., "The Variational Theory of the Plane Motion of the Extensible Elastica", Int. J. Engng.Sci. 4 (1966) 433-450.

Wempner, G., "Mechanics of Deformable Solids," Wempner, 1991.

Whitman, A.B. and C.N. DeSilva, "Dynamics and Stability of Elastic Cosserat Curves", Int. J. Solids Structures 6 (1970) 411-422.

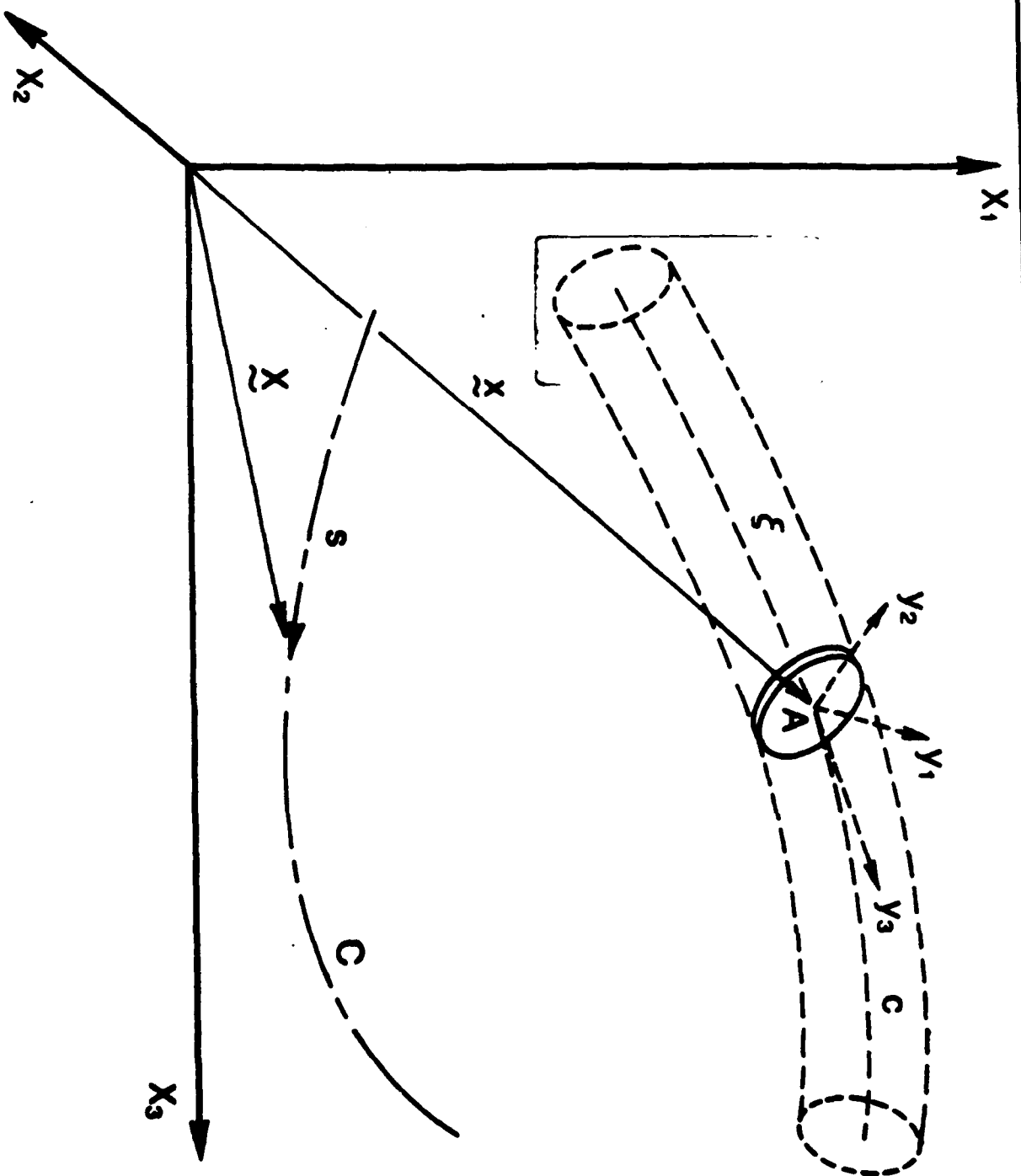


Fig. 1